Math 2J
Lecture 9 - 10/17/12

William Holmes

Recap

- The ***determinant*** is a function that assigns a number to a **square** matrix "A" so that

- $$det(A) = 0 \Leftrightarrow A \text{ is singular}$$
  $$det(A) \neq 0 \Leftrightarrow A \text{ is non-singular}$$

Computing a Determinant

- Can be computed by ‘method of diagonals’ for 2x2 and 3x3.
- For larger matrices, expansion by minors is necessary.

Properties

$$det(A \cdot B) = det(A) \cdot det(B)$$

$$det(A^T) = det(A)$$

- You must take care though.

$$det(A + B) \neq det(A) + det(B)$$

One more property

- If 'A' is invertible, then $det(A^{-1}) = \frac{1}{det(A)}$
- Proof

$$A^{-1} \cdot A = \mathbb{I} \qquad \mathbb{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\updownarrow$$

$$det(A^{-1}) \cdot det(A) = det(A^{-1} \cdot A) = det(\mathbb{I}) = 1$$

$$\updownarrow$$

$$det(A^{-1}) = \frac{1}{det(A)}$$

Special Matrices

- Diagonal matrix

$$det\left(\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}\right) = d_1 \cdot d_2 \cdot d_3$$

Special Matrices

- Triangular matrix.

$$det\left(\begin{bmatrix} d_1 & a & b \\ 0 & d_2 & c \\ 0 & 0 & d_3 \end{bmatrix}\right) = d_1 \cdot d_2 \cdot d_3$$

$$det\left(\begin{bmatrix} d_1 & 0 & 0 \\ a & d_2 & 0 \\ b & c & d_3 \end{bmatrix}\right) = d_1 \cdot d_2 \cdot d_3$$

The determinant and Gaussian Elimination

- Switch two rows - multiply the determinant by (-1)
- Multiply a row by 'c' - multiply the determinant by (c)
- Perform R2 --> a*R1 + b*R2 - multiply the determinant by (b).

Collect Thoughts

Equivalent Statements

- A*x=b has a **single unique** solution for any ‘b’.
- ‘A’ has an inverse.
- ‘A’ is non-singular.
- ‘A’ can be turned into $\mathbb{I}$ using Gaussian Elimination - ‘A’ is ***row equivalent*** to $\mathbb{I}$
- $\det(A) \neq 0$

Solution methods

- Gaussian Elimination
- Matrix - vector / inverse method.

Determinant

- Used to determine if a matrix has an inverse
  - Alternatively, if a system has a solution.

Computing the Determinant

- For a 2x2 or 3x3, use method of diagonals.
- For an nxn, use expansion by minors on a row.
- Use expansion by minors on a column.
- Use row reduction to simplify first!
  - Can be used to get a row or column of mostly zeros!

Determinant and System Solutions

- Not only can the determinant be used to see if a system has a solution.
- It can be directly used to compute the solution.
    - ***<u>Cramer's Rule</u>***

Methods of Solving a Linear System

- Gaussian Elimination
  - Best if you need to solve a system only once.
- Matrix - vector method.
  - Best if you need to solve a system many times.
- Cramer's Rule
  - Best if you need only one component of a solution, say x_5.